Reinforcement Learning

Study Notes: From the basics to PPO and GRPO

- What is the reinforcement learning?
- What are PPO and GRPO?
- How is RL utilized in LLMs?

Xin Zhang

Reinforcement Learning

2025 Turing Award

deepseek

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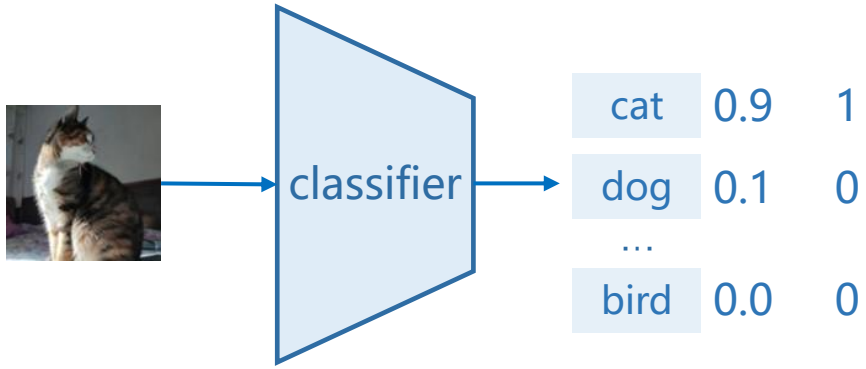
2016 AlphaGo

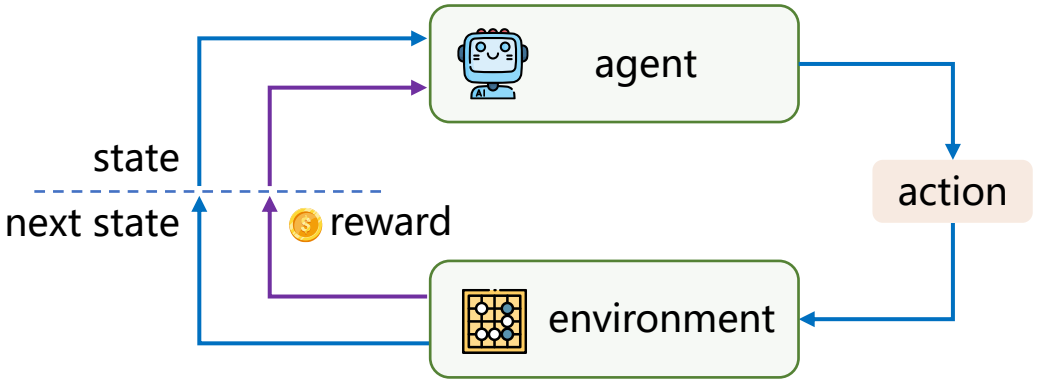
What is Reinforcement Learning



Supervised Learning

labeled data

Find a model to fit the mapping.

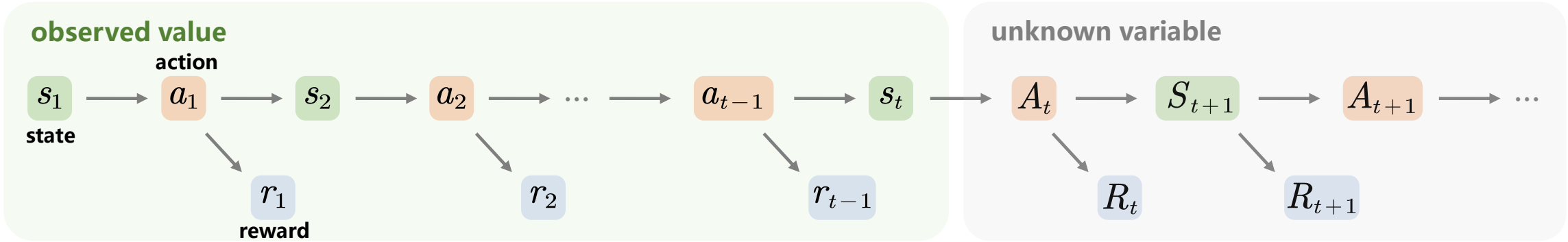


Reinforce Learning

Interaction

Find a policy to maximize the reward.

Markov Decision Process



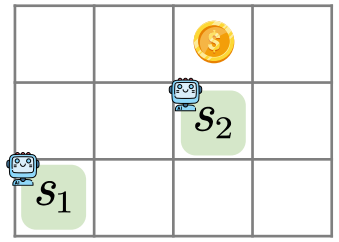
Return $U_1 = R_1 + \gamma R_2 + \cdots + \gamma^{t-2} R_{t-1} + \gamma^{t-1} R_t + \gamma^t R_{t+1} + \cdots + \gamma^{n-1} R_n$

Return (next state) $U_2 = R_2 + \cdots + \gamma^{t-3} R_{t-1} + \gamma^{t-2} R_t + \gamma^{t-1} R_{t+1} + \cdots + \gamma^{n-2} R_n$

Discounted Return

Immediate rewards | Future rewards

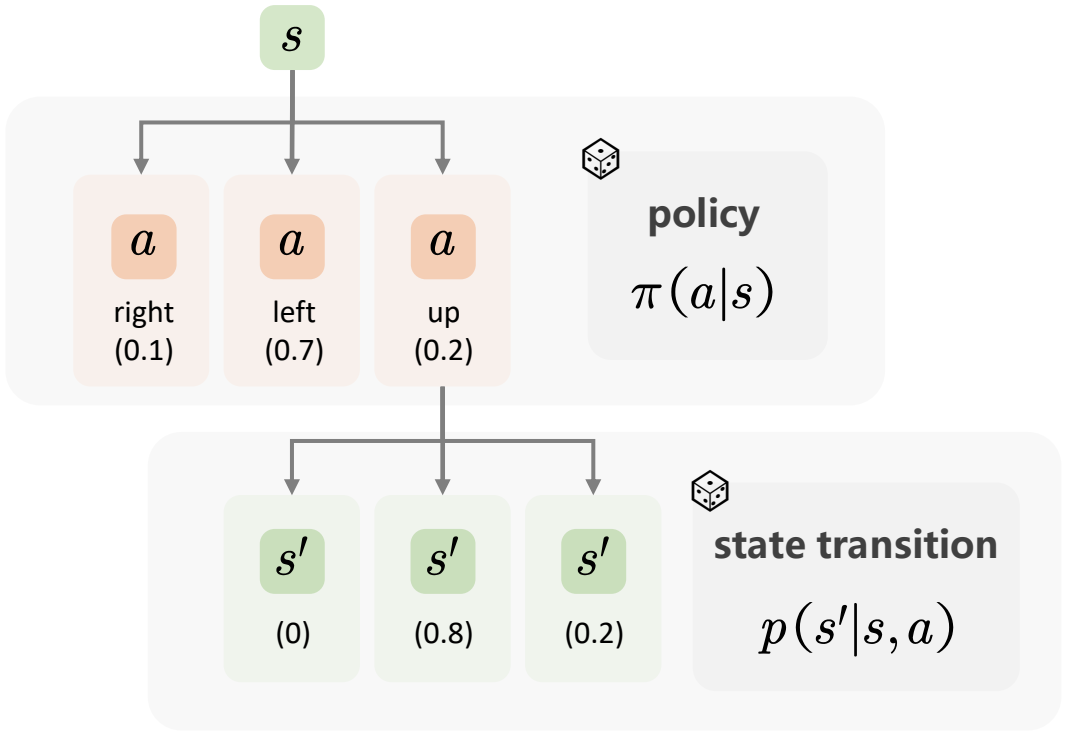
$$U_t = \sum_{k=t}^{n} \gamma^{k-t} R_k \qquad U_t = R_t + \gamma U_{t+1}$$



$$V(s_2) > V(s_1)$$

$$Q(s_2, up) > Q(s_2, down)$$

Randomness and Expectation



Expectation

$$\mathbb{E}_{x\sim \mathrm{p}(x)}[h(x)] = \sum_{x\in\mathcal{X}} p\,(x)\cdot h(x)$$

state value Bellman Equation

$$\begin{aligned}
V_\pi(s) &= \mathbb{E}_{A_t,S_{t+1}\ldots}[U_t|S_t=s] \\
&= \mathbb{E}_{A_t,S_{t+1}\ldots}[R_t + \gamma U_{t+1}|S_t=s] \\
&= \mathbb{E}_{A_t,S_{t+1}\ldots}[R_t + \gamma V_\pi(S_{t+1})|S_t=s] \\
&= \sum_{a\in A}\pi\,(a\mid s)\left(r(s,a) + \gamma\sum_{s'\in S} p\,(s'\mid s,a)V^\pi(s')\right)
\end{aligned}$$

(R_t: A_t; U_{t+1}: A_t, S_{t+1})

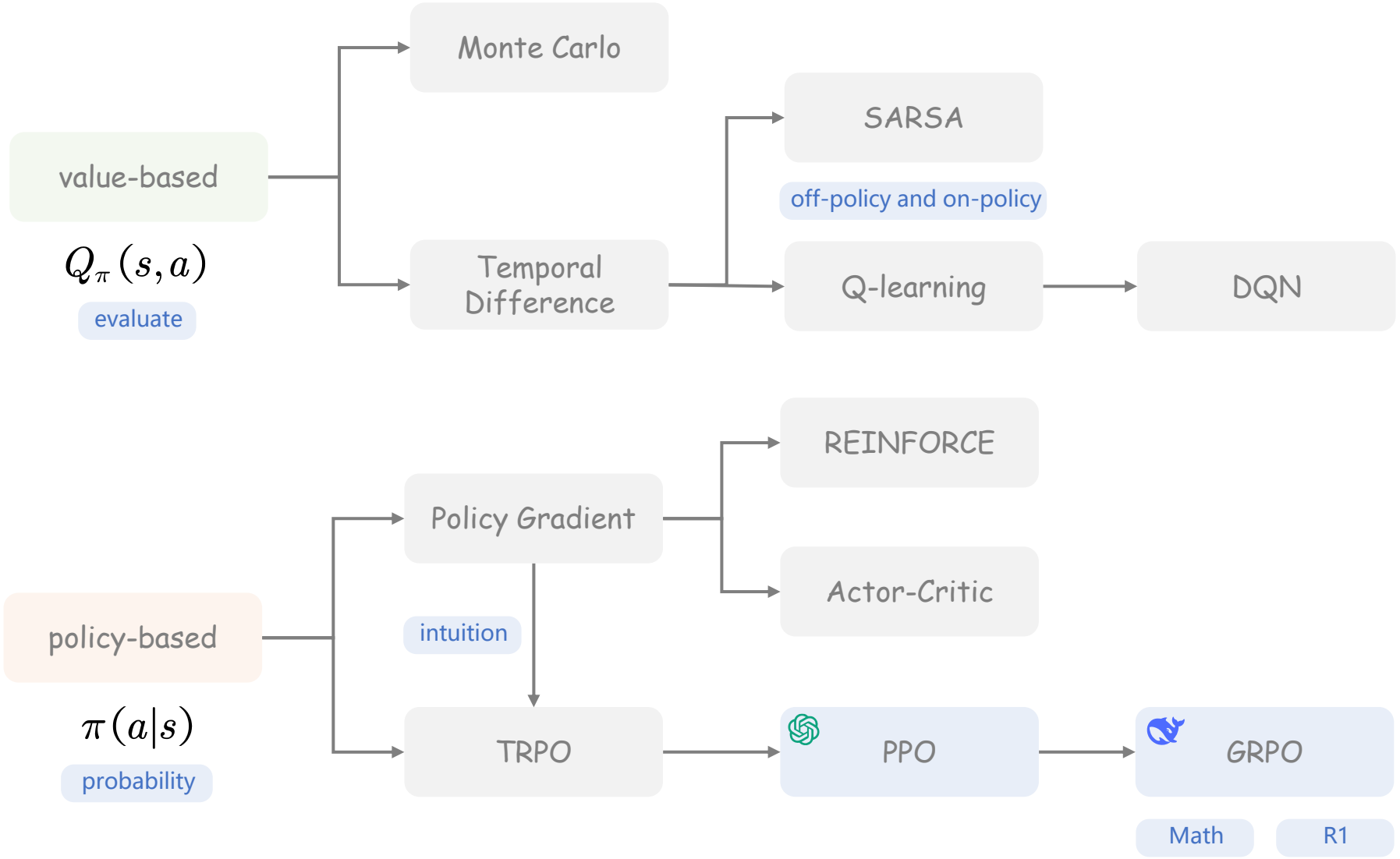
action value Bellman Equation

$$\begin{aligned}
Q_\pi(s,a) &= \mathbb{E}_{S_{t+1}\ldots}[U_t|S_t=s,A_t=a] \\
&= \mathbb{E}_{S_{t+1}\ldots}[R_t + \gamma U_{t+1})|S_t=s,A_t=a] \\
&= \mathbb{E}_{S_{t+1}\ldots}[R_t + \gamma Q_\pi(S_{t+1},A_{t+1})\mid S_t=s,A_t=a] \\
&= r(s,a) + \gamma\sum_{s'\in S} p\,(s'\mid s,a)\sum_{a'\in A}\pi\,(a'\mid s')Q_\pi(s',a')
\end{aligned}$$

(U_{t+1}: S_{t+1}, A_{t+1})

relationship $V_\pi(s) = \sum_{a\in A}\pi\,(a|s)Q_\pi(s,a)$

The Top-Down Framework

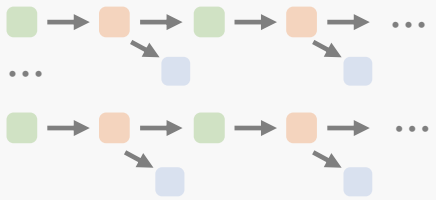


Temporal Difference

How to estimate $Q_\pi(s,a)$

$$Q_\pi(s,a) = \mathbb{E}_{S_{t+1}\cdots}[R_t + \gamma Q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a]$$

1 Monte Carlo multi-step variance ↑ bias ↓

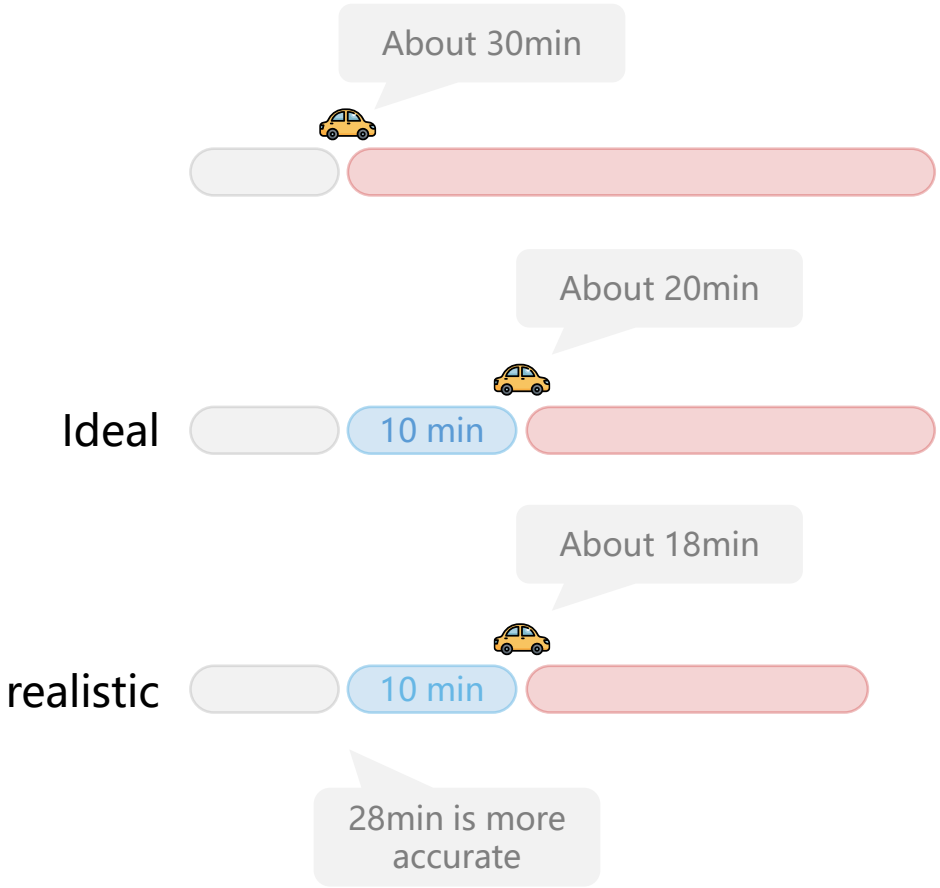


$$Q_\pi(s,a) \approx \frac{1}{N}\sum_{i=1}^{N} U^{(i)}$$

2 Temporal Difference one-step variance ↓ bias ↑

$$Q_\pi(s_t,a_t) \approx \underbrace{r_t + \gamma Q_\pi(s_{t+1}, a_{t+1})}_{\textbf{TD target}}$$

$$Q_\pi(s_t,a_t) \leftarrow Q_\pi(s_t,a_t) + \alpha[r_t + \gamma Q_\pi(s_{t+1},a_{t+1}) - Q_\pi(s_t,a_t)]$$



SARSA and Q-learning

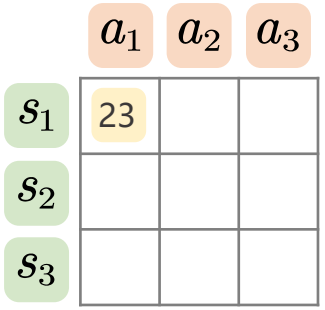
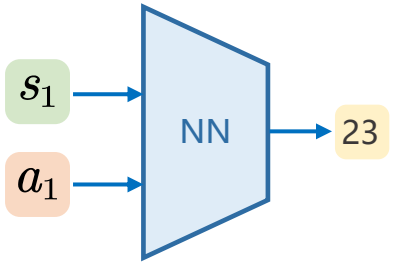


Table Query

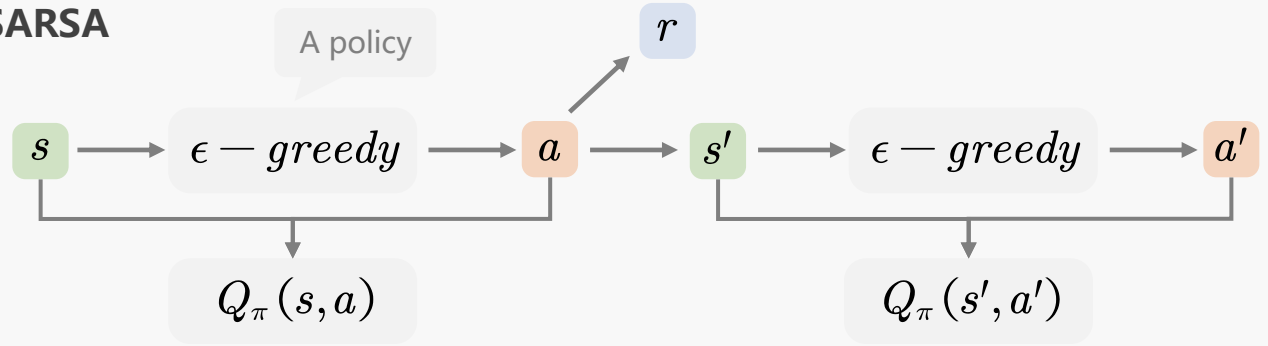
(SARSA, Q-Learning)



Neural Network

(DQN)

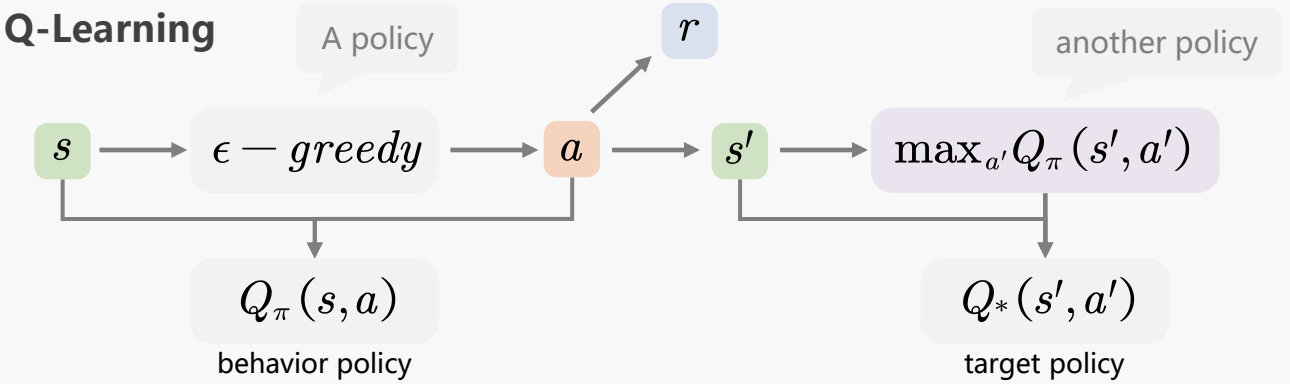
SARSA



obtain $Q_\pi(s,a)$

on-policy

Q-Learning



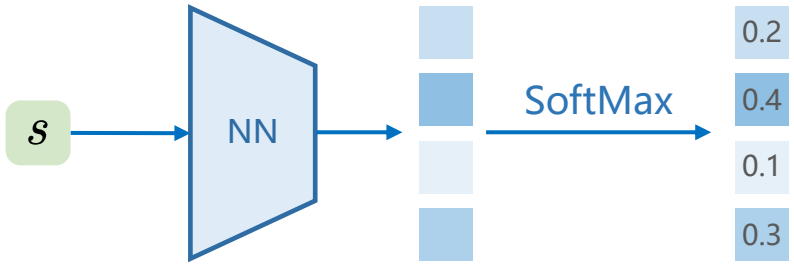
obtain $Q_*(s,a)$

off-policy

Table Update: $Q_\pi(s_t,a_t) \leftarrow Q_\pi(s_t,a_t) + \alpha[r_t + \gamma Q_\pi(s_{t+1},a_{t+1}) - Q_\pi(s_t,a_t)]$

NN Update: $Loss = \frac{1}{2}(r + \gamma Q_\pi(s',a') - Q_\pi(s,a))^2$

Policy Gradient



$$\pi(\cdot|s;\theta)$$

Maximize the return

$$J(\boldsymbol{\theta}) = \mathbb{E}_S[V_\pi(S)] = \mathbb{E}_{\tau \sim p_{\boldsymbol{\theta}}(\tau)}\left[\sum_t \gamma^t r(s_t, a_t)\right]$$

Optimization $\max_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$

Gradient Ascent $\boldsymbol{\theta}_{new} \leftarrow \boldsymbol{\theta}_{old} + \beta \cdot \nabla_{\boldsymbol{\theta}} J(\theta_{old})$

Differentiate a composite function

$$\nabla \ln f(x) = \frac{\nabla f(x)}{f(x)}$$

$$\nabla_{\boldsymbol{\theta}} V_\pi(s) = \nabla_{\boldsymbol{\theta}} \sum_{a \in \mathcal{A}} \pi(a|s;\boldsymbol{\theta}) \cdot Q_\pi(s,a)$$

$$\approx \sum_{a \in \mathcal{A}} (\nabla_{\boldsymbol{\theta}} \pi(a|s;\boldsymbol{\theta}) \cdot Q_\pi(s,a))$$ Ignore one item

$$= \sum_{a \in \mathcal{A}} (\pi(a|s;\boldsymbol{\theta}) \cdot \nabla_{\boldsymbol{\theta}} \ln \pi(a|s;\boldsymbol{\theta}) \cdot Q_\pi(s,a))$$

$$= \mathbb{E}_{A \sim \pi(\cdot|s;\boldsymbol{\theta})}[\nabla_{\boldsymbol{\theta}} \ln \pi(A|s;\boldsymbol{\theta}) \cdot Q_\pi(s,A)]$$

random gradient

REINFORCE and Actor-Critic

Optimization $\max_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$

Gradient Ascent $\boldsymbol{\theta}_{new} \leftarrow \boldsymbol{\theta}_{old} + \beta \cdot \nabla_{\boldsymbol{\theta}} J(\theta_{old})$

$$\boldsymbol{\theta}_{new} \leftarrow \boldsymbol{\theta}_{old} + \beta \cdot Q_\pi(s,a) \cdot \nabla_{\boldsymbol{\theta}} \ln \pi(a|s;\boldsymbol{\theta}_{old})$$

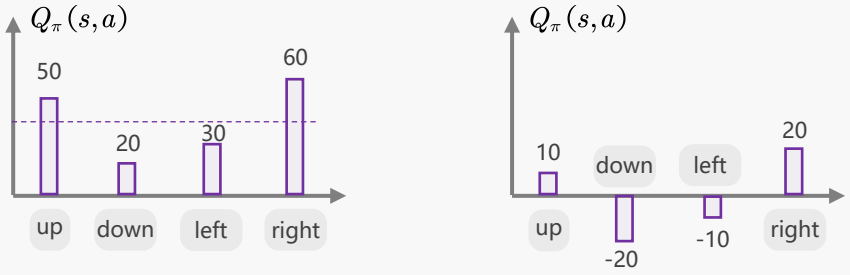
REINFORCE

$$u_t = \sum_{k=t}^{n} \gamma^{k-t} \cdot r_k$$

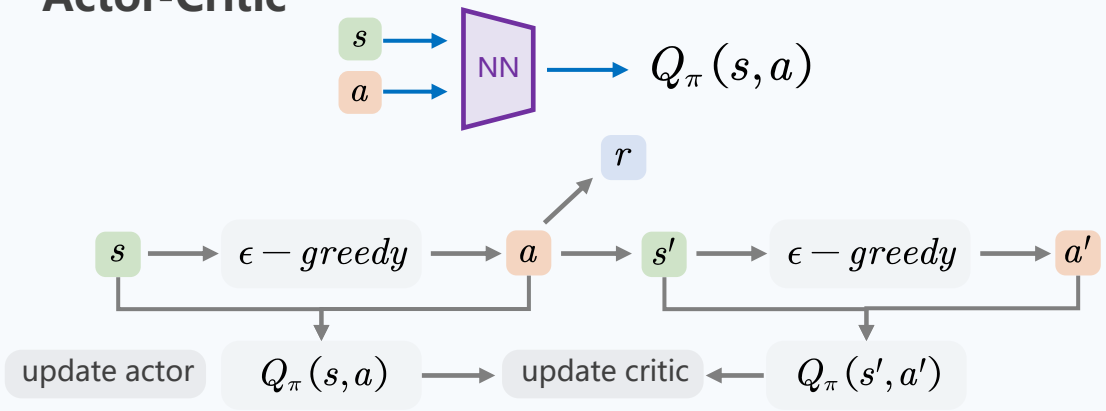
$$\boldsymbol{\theta}_{new} \leftarrow \boldsymbol{\theta}_{old} + \beta \cdot \sum_{t=1}^{n} \gamma^{t-1} \cdot u_t \cdot \nabla_{\boldsymbol{\theta}} \ln \pi(a_t|s_t;\boldsymbol{\theta}_{old})$$

Baseline

$$\mathbb{E}_S\left[_{A\sim\pi(\cdot|S;\boldsymbol{\theta})}\left[(Q_\pi(S,A) - b) \cdot \nabla_{\boldsymbol{\theta}} \ln \pi(A|S;\boldsymbol{\theta})\right]\right]$$



Actor-Critic



The Problem of Policy Gradient

Difficulty in setting the step size for updates.

Trust region

$$\|\boldsymbol{\theta}_{new} - \boldsymbol{\theta}_{old}\| \leqslant \Delta$$

A new policy is better

$$J(\boldsymbol{\theta}_{new}) - J(\boldsymbol{\theta}_{old}) > 0$$

Trust Region Policy Optimization

$J(\boldsymbol{\theta'}) - J(\boldsymbol{\theta})$ new policy old policy

$= J(\boldsymbol{\theta'}) - \mathbb{E}_{s_0 \sim p(s_0)}[V_{\pi_{\boldsymbol{\theta}}}(s_0)]$

$= J(\boldsymbol{\theta'}) - \mathbb{E}_{\tau \sim p_{\boldsymbol{\theta}'}(\tau)}[V_{\pi_{\boldsymbol{\theta}}}(s_0)]$ initial state unchanged

$= J(\boldsymbol{\theta'}) - \mathbb{E}_{\tau \sim p_{\boldsymbol{\theta}'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t V_{\pi_{\boldsymbol{\theta}}}(s_t) - \sum_{t=0}^{\infty} \gamma^{t+1} V_{\pi_{\boldsymbol{\theta}}}(s_{t+1})\right]$

$= \mathbb{E}_{\tau \sim p_{\boldsymbol{\theta}'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t)\right] + \mathbb{E}_{\tau \sim p_{\boldsymbol{\theta}'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t \left(\gamma V_{\pi_{\boldsymbol{\theta}}}(s_{t+1}) - V_{\pi_{\boldsymbol{\theta}}}(s_t)\right)\right]$

$= \mathbb{E}_{\tau \sim p_{\boldsymbol{\theta}'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t \left[r(s_t, a_t) + \gamma V_{\pi_{\boldsymbol{\theta}}}(s_{t+1}) - V_{\pi_{\boldsymbol{\theta}}}(s_t)\right]\right]$

$= \mathbb{E}_{\tau \sim p_{\boldsymbol{\theta}'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t \left[Q_{\pi_{\boldsymbol{\theta}}}(s_t, a_t) - V_{\pi_{\boldsymbol{\theta}}}(s_t)\right]\right]$

$= \mathbb{E}_{\tau \sim p_{\boldsymbol{\theta}'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t A_{\pi_{\boldsymbol{\theta}}}(s_t, a_t)\right]$

Advantage function

Importance Sampling

$\mathbb{E}_{x \sim p(x)}[f(x)]$

$= \int_x q(x) \frac{p(x)}{q(x)} f(x) dx = \mathbb{E}_{x \sim q(x)}\left[\frac{p(x)}{q(x)} f(x)\right]$

$\mathbb{E}_{\tau \sim p_{\boldsymbol{\theta}'}(\tau)}\left[\sum_{t=0}^{\infty} \gamma^t \cdot A_{\pi_{\boldsymbol{\theta}}}(s_t, a_t)\right]$

$= \mathbb{E}_{s_t \sim p_{\boldsymbol{\theta}'}(s_t)}\left[\mathbb{E}_{a_t \sim \pi_{\boldsymbol{\theta}'}(a_t|s_t)}\left[\gamma^t \cdot A_{\pi_{\boldsymbol{\theta}}}(s_t, a_t)\right]\right]$

Approximation 1: Ignoring the changes in the state distribution.

$= \mathbb{E}_{s_t \sim p_{\boldsymbol{\theta}}(s_t)}\left[\mathbb{E}_{a_t \sim \pi_{\boldsymbol{\theta}'}(a_t|s_t)}\left[\gamma^t \cdot A_{\pi_{\boldsymbol{\theta}}}(s_t, a_t)\right]\right]$

Approximation 2: Importance sampling.

$= \mathbb{E}_{s_t \sim p_{\theta}(s_t)}\left[\mathbb{E}_{a_t \sim \pi_{\theta}(a_t|s_t)}\left[\frac{\pi_{\theta'}(a_t|s_t)}{\pi_{\theta}(a_t|s_t)} \cdot \gamma^t \cdot A_{\pi_{\theta}}(s_t, a_t)\right]\right]$

TRPO

$\underset{\boldsymbol{\theta'}}{\operatorname{argmax}} \mathbb{E}_{s \sim v_{\boldsymbol{\theta}}, a \sim \pi_{\boldsymbol{\theta}}(\cdot|s)}\left[\frac{\pi_{\boldsymbol{\theta'}}(a, s)}{\pi_{\boldsymbol{\theta}}(a, s)} \cdot A_{\pi_{\boldsymbol{\theta}}}(s, a)\right]$

$s.t.\ D_{KL}(\pi_{\boldsymbol{\theta}}(\cdot|s) \ || \ \pi_{\boldsymbol{\theta'}}(\cdot|s))$

Proximal Policy Optimization

TRPO

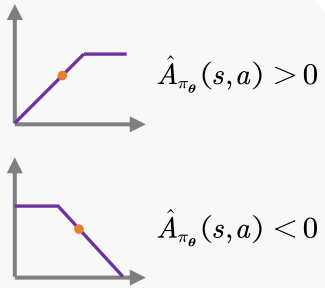
$$\underset{\theta'}{\operatorname{argmax}} \mathbb{E}_{s\sim v_\theta, a\sim\pi_\theta(\cdot|s)}\left[\frac{\pi_{\theta'}(a,s)}{\pi_\theta(a,s)}\cdot A_{\pi_\theta}(s,a)\right] \qquad s.t.\ D_{KL}(\pi_{\boldsymbol{\theta}}(\cdot|s)\ ||\ \pi_{\boldsymbol{\theta}'}(\cdot|s))$$

PPO-penalty

$$\underset{\boldsymbol{\theta}'}{\operatorname{argmax}} \mathbb{E}_{s\sim v_{\boldsymbol{\theta}}} \mathbb{E}_{a\sim\pi_{\boldsymbol{\theta}}(\cdot|s)}\left[\frac{\pi_{\boldsymbol{\theta}'}(a|s)}{\pi_{\boldsymbol{\theta}}(a|s)}\hat{A}_{\pi_{\boldsymbol{\theta}}}(s,a) - \beta D_{KL}[\pi_{\boldsymbol{\theta}}(\cdot|s)||\pi_{\boldsymbol{\theta}'}(\cdot|s)]\right] \qquad \begin{cases}\beta\leftarrow\beta/2 & if\ D_{KL}<\delta/1.5\\ \beta\leftarrow\beta\times 2 & if\ D_{KD}>\delta\times 1.5\end{cases}$$

PPO-clip

$$\underset{\boldsymbol{\theta}'}{\operatorname{argmax}} \mathbb{E}_{s\sim v_{\boldsymbol{\theta}}} \mathbb{E}_{a\sim\pi_{\boldsymbol{\theta}}(\cdot|s)}\left[\min\left(\frac{\pi_{\boldsymbol{\theta}'}(a|s)}{\pi_{\boldsymbol{\theta}}(a|s)}\hat{A}_{\pi_{\boldsymbol{\theta}}}(s,a),\ clip\left(\frac{\pi_{\boldsymbol{\theta}'}(a|s)}{\pi_{\boldsymbol{\theta}}(a|s)}, 1-\epsilon, 1+\epsilon\right)\hat{A}_{\pi_{\boldsymbol{\theta}}}(s,a)\right)\right]$$



Generalized Advantage Estimation

1 Monte Carlo multi-step variance ↑ bias ↓

2 Temporal Difference one-step variance ↓ bias ↑

➡ **Multi-Step Temporal Difference**

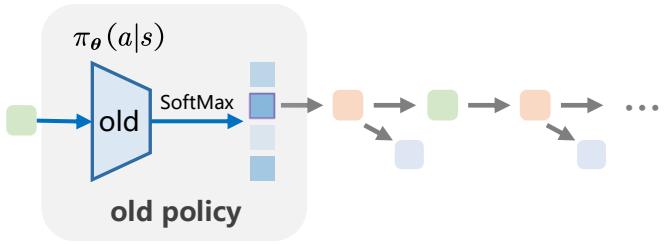
$$A_t^{(1)} = r_t + \gamma V(s_{t+1}) - V(s_t) \Rightarrow \delta_t \qquad A_{t+1}^{(1)} = r_{t+1} + \gamma V(s_{t+2}) - V(s_{t+1}) \Rightarrow \delta_{t+1}$$

$$A_t^{(2)} = r_t + \gamma r_{t+1} + \gamma^2 V(s_{t+2}) - V(s_t) \Rightarrow \delta_t + \gamma\, \delta_{t+1}$$

$$A_t^{(k)} = r_t + \gamma r_{t+1} + \cdots + \gamma^{k-1} r_{t+k-1} + \gamma^k V(s_k) - V(s_t) \Rightarrow \delta_t + \gamma\, \delta_{t+1} + \cdots + \gamma^{k-1}\, \delta_{t+k-1}$$

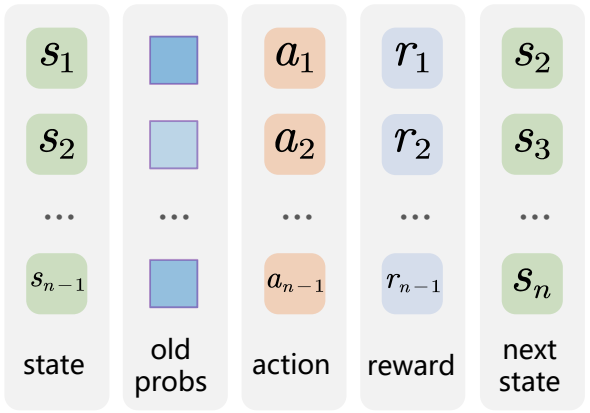
$$\begin{aligned} A_t^{GAE} &= (1-\lambda)\left(A_t^{(1)} + \lambda A_t^{(2)} + \lambda^2 A_t^{(3)} + \cdots\right) \\ &= (1-\lambda)\left(\delta_t(1+\lambda+\lambda^2+\cdots) + \gamma\delta_{t+1}(\lambda+\lambda^2+\cdots) + \cdots\right) \\ &= (1-\lambda)\left(\delta_t \frac{1}{1-\lambda} + \gamma\delta_{t+1}\frac{\lambda}{1-\lambda} + \gamma^2\delta_{t+2}\frac{\lambda^2}{1-\lambda} + \cdots\right) = \sum_{l=0} (\gamma\lambda)^l \delta_{t+l} \end{aligned}$$

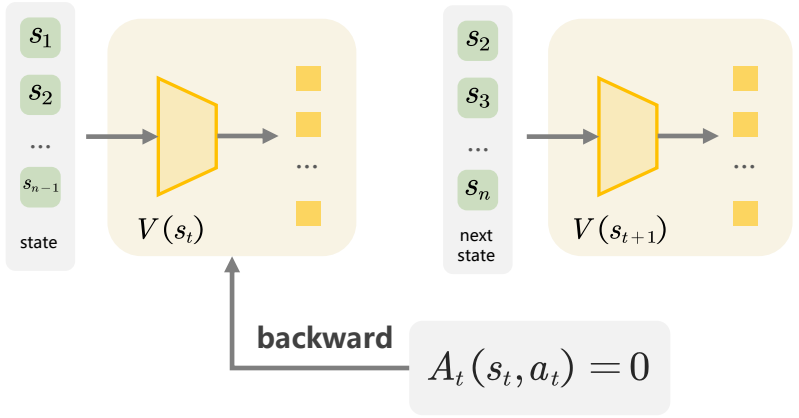
Training Process of PPO



PPO-clip

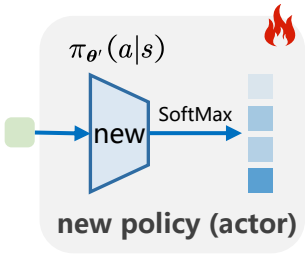
$$\underset{\theta'}{\operatorname{argmax}}\, \mathbb{E}_{s\sim v_\theta}\mathbb{E}_{a\sim\pi_\theta(\cdot|s)}\left[\min\left(\frac{\pi_{\theta'}(a|s)}{\pi_\theta(a|s)}A^{GAE}{}_{\pi_\theta}(s,a),\ clip\left(\frac{\pi_{\theta'}(a|s)}{\pi_\theta(a|s)},1-\epsilon,1+\epsilon\right)A^{GAE}{}_{\pi_\theta}(s,a)\right)\right]$$

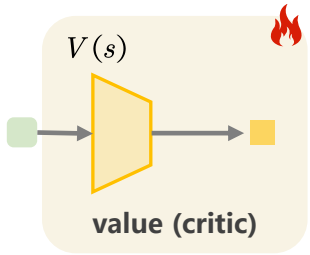


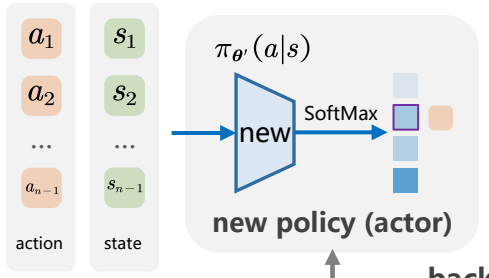


$$A_t(s_t,a_t) = r_1 + r_2 + \ldots + r_{n-1} \text{ (reward)} + \gamma V(s_{t+1}) - V(s_t)$$

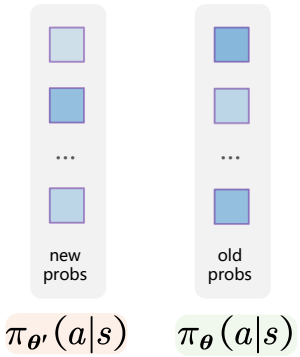
$$A^{GAE}{}_t(s_t,a_t) = \sum_{l=0}(\gamma\lambda)^l\delta_{t+l}$$



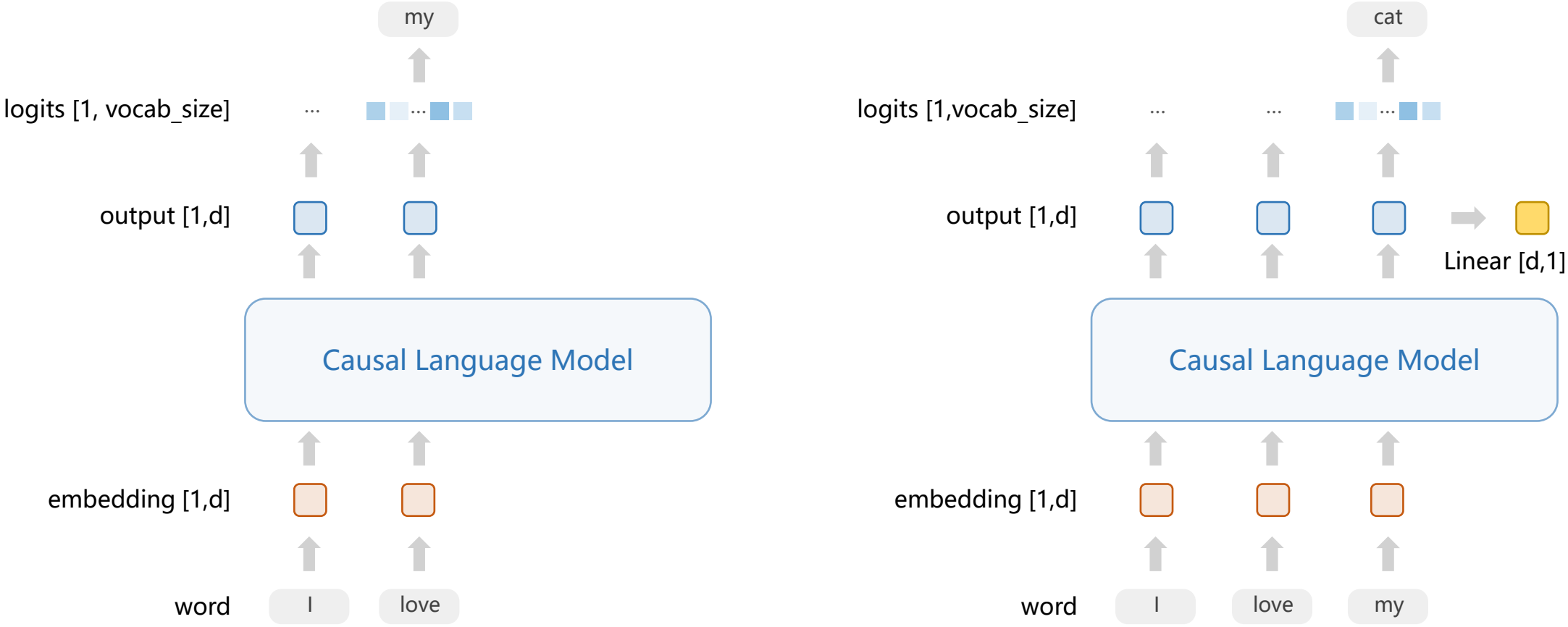




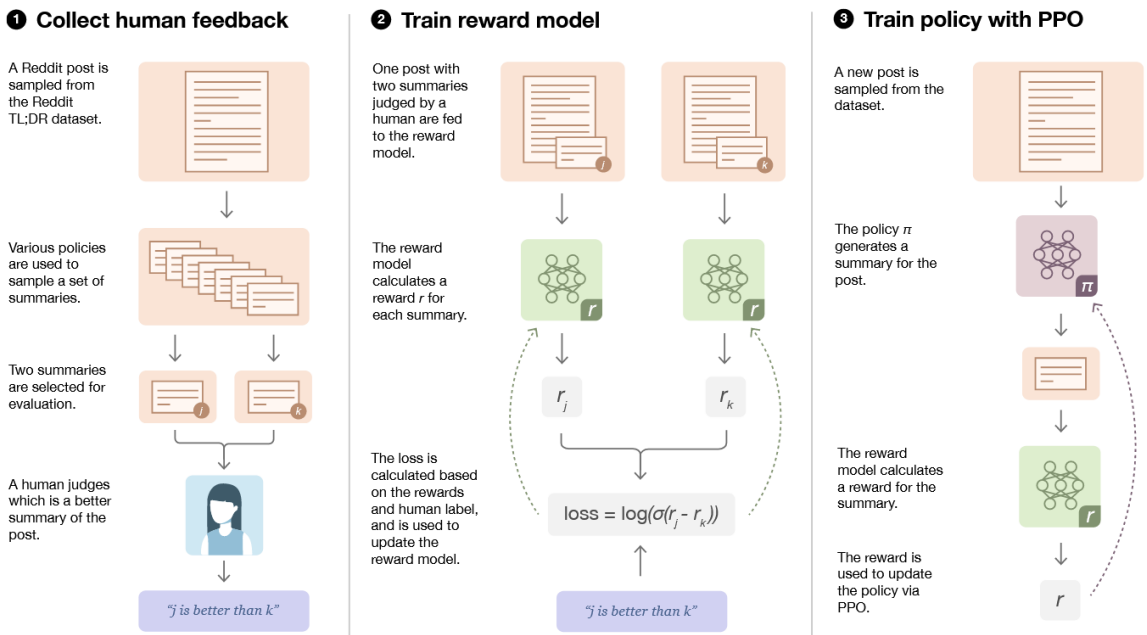
PPO

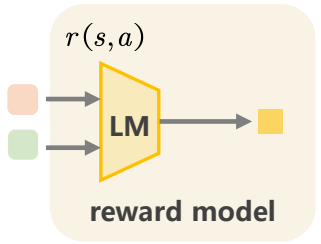


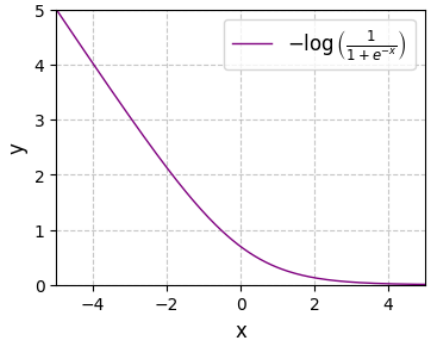
Large Language Model



How To Train A Reward Model







Bradley-Terry

	win	lose
A vs. B	8	4
A vs. C	3	5
B vs. C	2	3

$A:\pi_a$

$B:\pi_b$

$C:\pi_c$

$$P_{A>B}=\frac{\pi_a}{\pi_a+\pi_b}$$

Maximum Likelihood Estimation

$$L=\prod_{i\neq j}\left(\frac{\pi_i}{\pi_i+\pi_j}\right)^{n_{ij}}$$

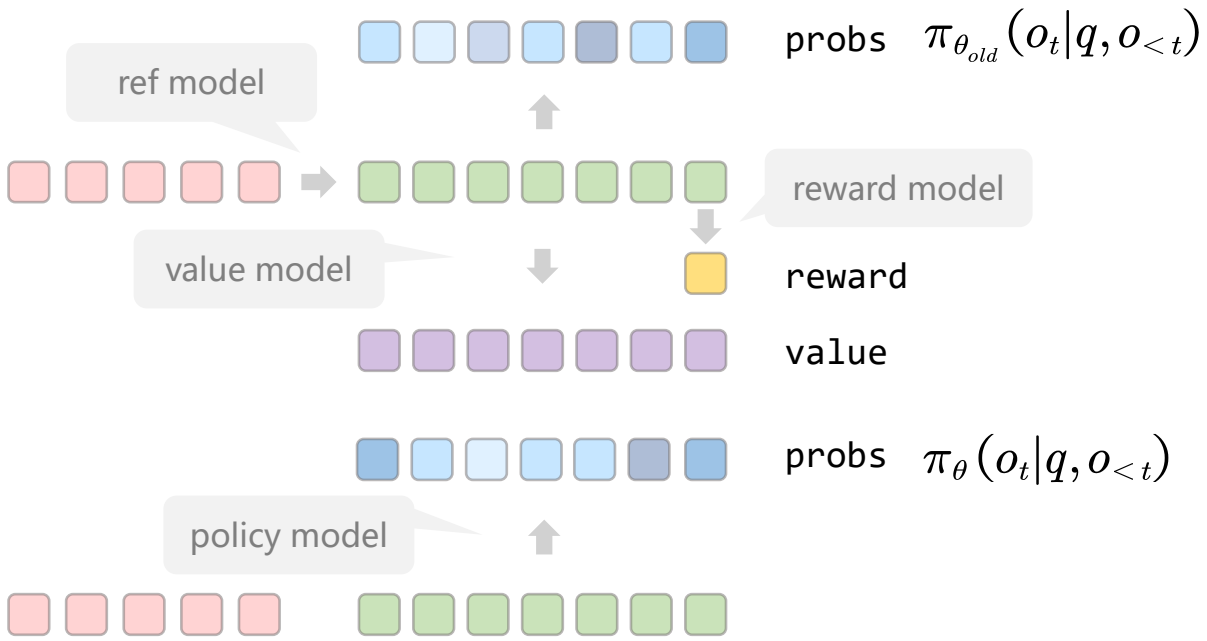
$$L=\prod\left(\frac{\exp(r_{\boldsymbol{\theta}}(x,yes))}{\exp(r_{\boldsymbol{\theta}}(x,yes))+\exp(r_{\boldsymbol{\theta}}(x,no))}\right)$$

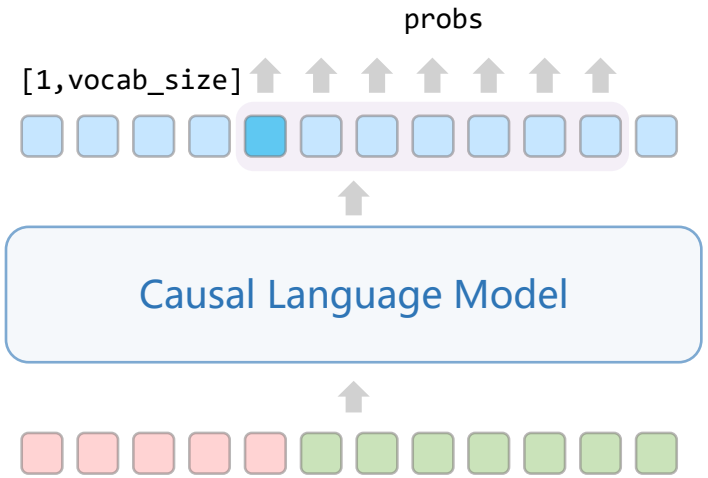
$$\ln L=\ln\sum\frac{1}{1+\exp(r_{\boldsymbol{\theta}}(x,no)-r_{\boldsymbol{\theta}}(x,yes))}\quad\sigma=\frac{1}{1+e^{-x}}$$

$$Loss=\mathbb{E}\left[\ln\sigma(r_{\boldsymbol{\theta}}(x,yes)-r_{\boldsymbol{\theta}}(x,no))\right]$$

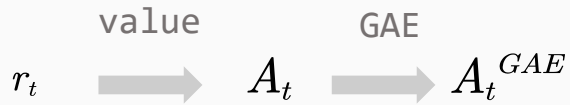
Training Process of PPO in LLM

$$\mathcal{J}_{PPO}(\theta) = \mathbb{E}\left[q \sim P(Q), o \sim \pi_{\theta_{old}}(O|q)\right] \frac{1}{o} \sum_{t=1}^{|o|} \min\left[\frac{\pi_\theta(o_t|q,o_{<t})}{\pi_{\theta_{old}}(o_t|q,o_{<t})} A_t, clip\left(\frac{\pi_\theta(o_t|q,o_{<t})}{\pi_{\theta_{old}}(o_t|q,o_{<t})}, 1-\epsilon, 1+\epsilon\right) A_t\right]$$



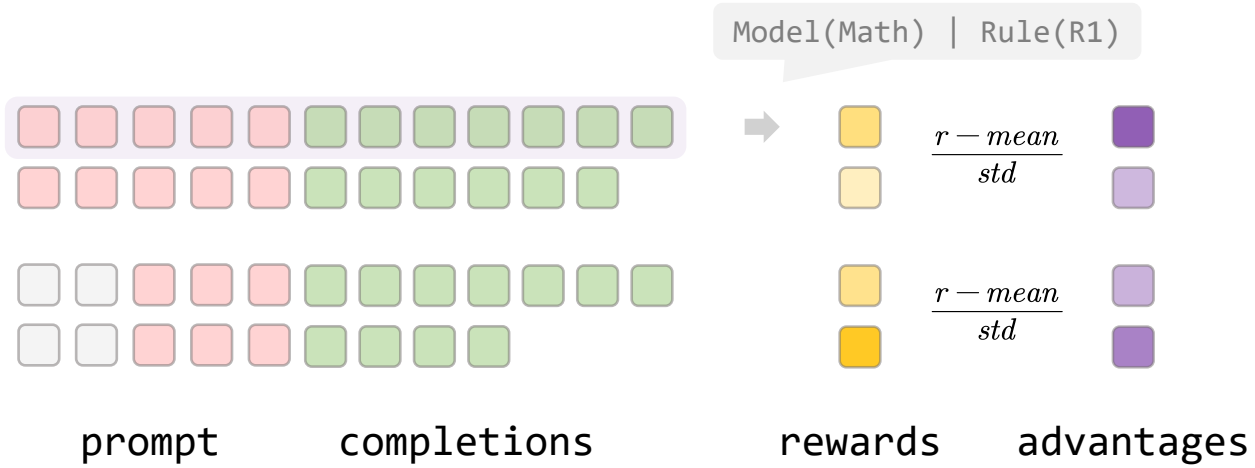


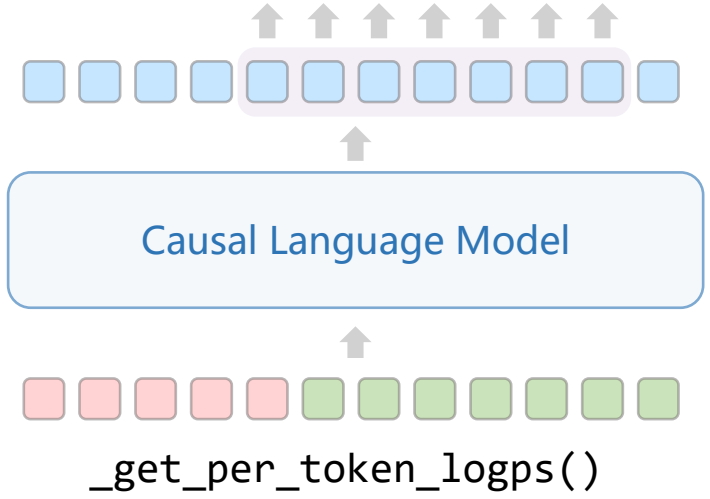
$$r_t = \underbrace{r_\varphi(q, o_{\leqslant} t)}_{\text{reward}} - \underbrace{\beta \ln \frac{\pi_\theta(o_t|q,o_{<t})}{\pi_{\theta_{old}}(o_t|q,o_{<t})}}_{\text{KL}}$$

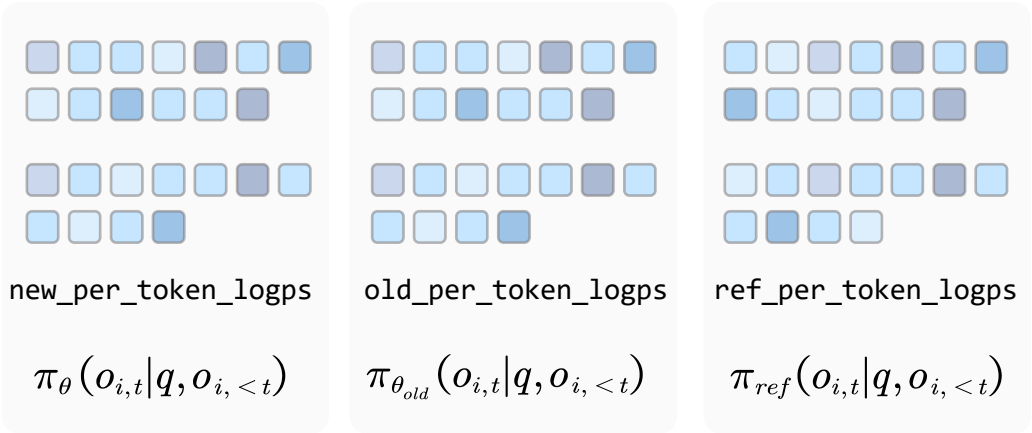


$$A_t = r_t + \gamma V_{t-1} - V_t$$

Group Relative Policy Optimization





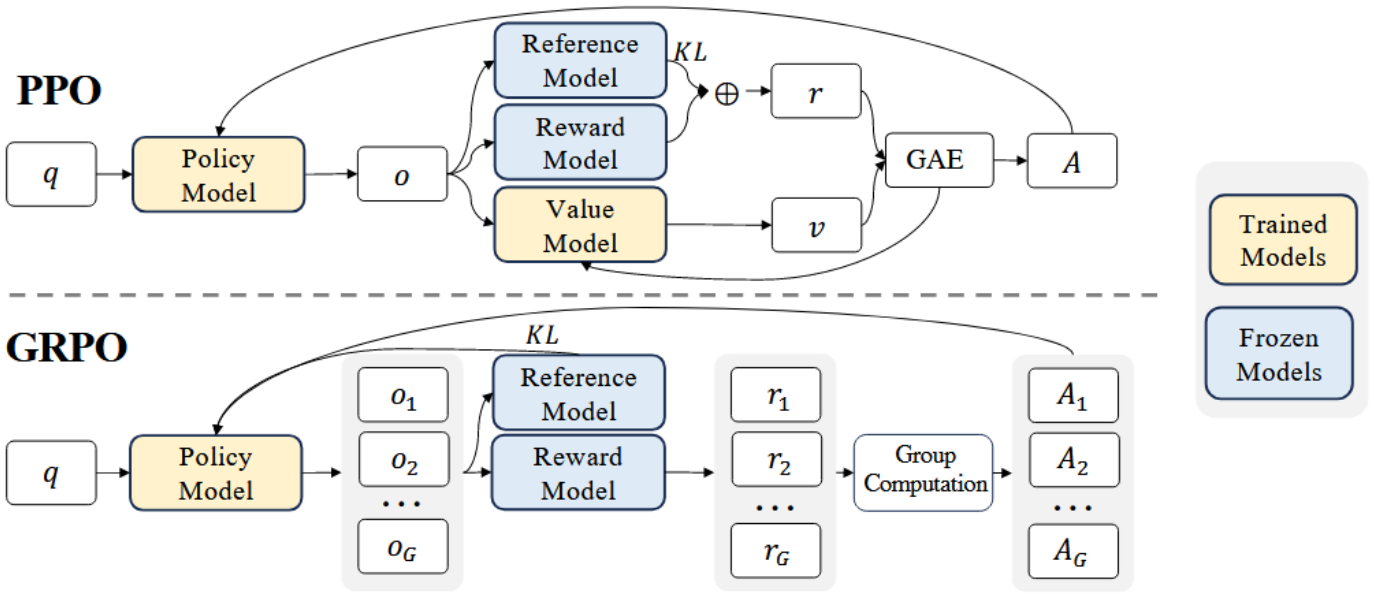


$$\mathcal{J}_{GRPO}(\theta) = \mathbb{E}\left[q \sim \mathrm{P}(Q), \{o_i\}_{i=1}^{G} \sim \pi_{\theta_{old}}(O|q)\right]$$

$$\frac{1}{G}\sum_{i=i}^{G}\frac{1}{|o_i|}\sum_{t=1}^{o_i}\left\{\min\left[\frac{\pi_\theta}{\pi_{\theta_{old}}}\hat{A}_{i,t}, clip\left(\frac{\pi_\theta}{\pi_{\theta_{old}}}, 1-\epsilon, 1+\epsilon\right)\hat{A}_{i,t}\right] - \beta\mathbb{D}_{KL}\left[\pi_\theta||\pi_{ref}\right]\right\}$$

$$\mathbb{D}_{KL}\left[\pi_\theta||\pi_{ref}\right] = \frac{\pi_{ref}}{\pi_\theta} - \log\frac{\pi_{ref}}{\pi_\theta} - 1$$

Review the PPO and GRPO



PPO

policy model	ref model
reward model	value model

GRPO

policy model	ref model
reward model	

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